The State of the Jacobian Conjectures

Structure, verification, and scope, 72 hours after the counterexample

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Abstract

On July 19, 2026, L. Alpöge announced an explicit polynomial map $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ with $\det JF \equiv -2$ sending three points to one, refuting the Jacobian Conjecture for $n \geq 3$. Within 48 hours, families covering every generic fiber degree ≥ 3 (Gallagher), a parametric factory in generalized weights (Zyskind–Sol), and a dimension-4 example appeared. We record what is now *proved*, at honest strength: every known counterexample is a torus-equivariant twisted lift governed by a single bracket identity (the *master equation*); the shadow ramification is universally $c\,m^2$; étale self-maps of $\mathbb{C}^3$ form a monoid realizing exactly the degrees $\{1\}\cup\{3,4,\dots\}$; and, at the two founding examples, first-order deformation theory shows the torus symmetry is locally forced under stated caps. **Scope:** all results concern the *quasi-torus* class; nothing here decides the plane case JC_2, which remains the Jacobian Conjecture.

1 The counterexample and certificates

Write $u = 1 + xy$. Alpöge's map is $F = (u^3 z + y^2 u(4 + 3xy),\ y + 3xu^2 z + 3xy^2(4 + 3xy),\ 2x - 3x^2 y - x^3 z)$.

Theorem 1 (Alpöge 2026; verified [C1–C4])**.** *$\det JF \equiv -2$, and $F(0,0,-\frac{1}{4}) = F(1,-\frac{3}{2},\frac{13}{2}) = F(-1,\frac{3}{2},\frac{13}{2}) = (-\frac{1}{4},0,0)$. Hence JC_n is false for all $n \geq 3$.*

An independent certificate found in-session before seeing the announcement: $F(1,-2,9) = F(-\frac{1}{3},4,27) = F(-\frac{2}{3},-\frac{1}{2},-\frac{9}{8}) = (-1,1,-1)$. Both are exact rational arithmetic; the determinant identity was checked by two independent implementations, including from-scratch exact arithmetic.

2 The mechanism: descent and the master equation

F is equivariant for $\lambda \cdot (x,y,z) = (\lambda^{-1}x, \lambda y, \lambda^2 z)$, with invariants $t = xy$, $w = x^2 z$. Setting $m = F_3/x$, $\tilde{g} = xF_2$, $\tilde{h} = x^2 F_1$ (evaluated at $x = 1$), the map descends to the plane as $G = (m\tilde{g},\ m^2\tilde{h})$ with the x-direction multiplied by m.

Theorem 2 (master equation; verified [C5] on all tested maps)**.** *$\det JF$ is constant $\neq 0$ iff $m\{\tilde{g},\tilde{h}\} + 2\tilde{h}\{\tilde{g},m\} + \tilde{g}\{m,\tilde{h}\} = c \neq 0$, where $\{p,q\} = p_t q_w - p_w q_t$; then $\det JF = -c$ and $\operatorname{jac} G = c\,m^2$ identically.*

*Model-assisted throughout; all claims machine-verified where so marked. Verifier: `verifier.py` at jacobian-conjectures.com; exact symbolic arithmetic, no floating point.

Proof sketch. Chain rule through $(x,y,z)\mapsto(x,t,w)$: $\det JF = \operatorname{jac} G/m^2$; expand $\operatorname{jac}(m\tilde g, m^2\tilde h) = m^2\big(m\{\tilde g,\tilde h\} + 2\tilde h\{\tilde g, m\} + \tilde g\{m,\tilde h\}\big)$. $\square$

Consequences: all ramification of the shadow lies on the single crushed line $\{m=0\}$ with multiplicity two, *at every degree*; higher degree means more sheets over one crease, never more creases. Alpöge satisfies the equation with $c=2$; Gallagher's degree-4 map with $c=-1$; the Zyskind–Sol factory in generalized weights $A = 1+xy^m$. Gallagher's seed form (p with $p(0)=0$, $p(1)=-c$, $\int_0^1 p = 0$; fiber degree $\deg p+1$) is an integral-form cycle condition: the potential $P=\int p$ returns to zero — counterexamples correspond to zero-net-work cycles.

3 The monoid and the degree spectrum

Theorem 3 (degree spectrum)**.** *Generic fiber degrees of étale polynomial self-maps of $\mathbb{C}^3$ are exactly $\{1\}\cup\{3,4,5,\dots\}$.*

Proof. No degree 2: an index-2 field extension is Galois; by Campbell (1973) a Keller map with Galois extension is an automorphism. *All $n\geq 3$:* Gallagher's seed family (verified at degree 4, [C9]); degree 3 is Alpöge's map. Composition (degrees multiply; multiplier cocycle $m_{12} = m_2\cdot(m_1\circ G_2)$, verified [C8]) gives closure. $\square$

Thus 3 is the least possible counterexample degree: the first non-Galois opportunity, and it was taken.

4 Walls around the plane, and closed universes

Proposition 4 (quasi-torus wall; two lines)**.** *In dimension 2 the invariant base is one-dimensional and the master equation degenerates to $g'(t)=c$: the twist mechanism produces only automorphisms. No quasi-torus counterexample exists in $\mathbb{C}^2$.*

Together with Moh ($\deg\leq 100$) and the prime-extension exclusion (arXiv:2407.13795), these are three *partial* walls; none decides JC_2. Classically, a *proper* étale self-map of $\mathbb{C}^n$ is a trivial covering: in a closed universe the conjecture is true, and every counterexample is an open-system phenomenon — its fold exiled through the non-properness set. On the honest torus the phenomenon is trivial ($x\mapsto x^2$, log-Jacobian 2).

Definition 5 (faked torus)**.** *F fakes a torus if over the complement of its Jelonek set A_F it restricts to a covering of degree ≥ 2; for all known examples A_F lies in a hyperplane and $\pi_1(\mathbb{C}^3\setminus A_F)\cong\mathbb{Z}$ plays the role of the missing $\mathbb{C}^*$.* JC_2 *is equivalent to: no Keller map of $\mathbb{C}^2$ fakes a torus or otherwise fails properness with monodromy — the non-toric route is precisely what remains open.*

5 Local rigidity (computational, capped)

At F (deformations of degree ≤ 7, two primes): the tangent space to the Keller locus has dimension 33; joint composition moves (with cross-cancellation between $JF\cdot V$ and $W\circ F$) plus equivariant deformations explain 32; the residual direction, pure torus-weight offset $+3$, is obstructed at second order, cross-terms included. At Gallagher's F_4: $\dim T = 11$, fully explained after the deep conjugation audit. *Within stated caps, the torus symmetry is locally forced at both founding examples.* Caveats: degree caps, two basepoints, modular arithmetic (two primes), local only.

6 Verification artifacts

`verifier.py`: nine checks (C1–C9), exact symbolic arithmetic, independent reimplementation of the determinant, exit nonzero on any failure. `formal/JacobianTorus.lean`: Lean 4/mathlib draft — torus triviality, collision certificates (`norm_num`), determinant as ring identity, the dim-2 wall; compile pending toolchain; `sorry` count is a published status line, not a secret.

7 Open problems

Question 6 (generation)**.** *Is every étale self-map of $\mathbb{C}^3$ a composition of automorphisms and quasi-torus lifts?*

Question 7 (equivariance forced)**.** *Do non-quasi-torus counterexamples exist in any dimension? (Local evidence says no; the evidence is two points deep.)*

Question 8 (quantization)**.** *Exhibit the explicit non-surjective endomorphism of the Weyl algebra A_3; compute its index and compare with the fiber degree.*

Question 9 (the plane)**.** JC_2*. Now with a structural target: rule the non-toric mechanisms in or out.*

Attribution and timeline. Alpöge (counterexample, 07-19); Gallagher (seed family, all degrees ≥ 3, 07-20); Zyskind–Sol (factory, generalized weights, 07-20); Zhang (consequences note, circulating); this project (master equation, spectrum lower half, monoid/cocycle, quasi-torus wall, rigidity computations, verification suite; timestamps in the public ledger). Corrections are recorded, never erased: the dustbin.